Alternating Convex Optimization for Truss Topology Design

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EE364b: Convex Optimization II Class Project

Introduction

- Desirable to have minimal flexing in a building, which is determined by how well the supporting truss structure is built
- A truss is defined by the size and shape of bars, and their attachment points (i.e., nodes) in some physical space $\mathcal{D} \subset \mathbf{R}^2$ (for a 2-D structure)
- Traditional approaches:
  - Discretize the space for nodes: $\hat{\mathcal{D}} \subset \mathbf{Z}^2$ (*e.g.*, Ben-Tal & Nemirovski)
  - Introduce complicated domain-specific heuristics (*e.g.*, Wang et. al.)

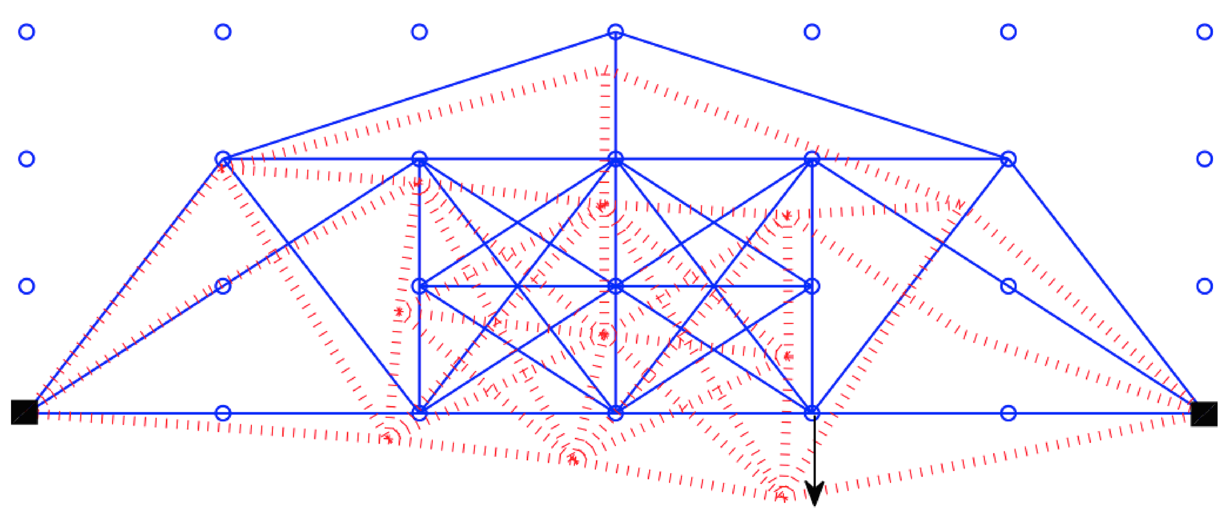

Figure 1: An example of a truss. Traditional approaches discretize the space for nodes, seen here as blue circles. The red dashed lines show the truss flexing under loading forces, seen here as a black arrow. Notice the fixed nodes depicted as solid black squares.

Truss Topology Optimization

Produce

- *Set of sized bars* $\mathcal{B}$ that constitute a truss
- *Set of attachment points or nodes* $\mathcal{X}$ for the bars

Given

- *Set of fixed nodes* $\mathcal{X}^{\text{fixed}} \subset \mathcal{X}$ representing the truss foundation
- *Set of loading forces* $\mathcal{F}$ that the truss is to be designed to support
- *Physical space* $\mathcal{D}$ that limits where $\mathcal{X}$ can be placed
- *Maximum allowable weight of truss* $W^{\max}$
- *Structural symmetry constraints*

To maximize the truss stiffness, which is related to the elastic stored energy $\Theta(\mathcal{F}, \mathcal{U})$, where $\mathcal{U}$ is the set of node deflections under load forces

Problem Formulation

The design variables for our truss optimization are:

- *Cross sectional areas* $a \in \mathbf{R}^m$, where $a_i \in \mathbf{R}$ is the area of the i^{th} bar
- *Coordinates* $x \in \mathbf{R}^{2n}$, where $x_j \in \mathbf{R}^2$ are the coordinates of the j^{th} node

Our problem data are:

- *Loading forces* $F \in \mathbf{R}^{2n}$, where $F_j \in \mathbf{R}^2$ is the load on the j^{th} node
- *Material densities* $\rho_1, \ldots, \rho_m \in \mathbf{R}$ of bars
- *Young's moduli* $E_1, \ldots, E_m \in \mathbf{R}$ characterizing the elasticities of the bars
- *Bar lengths* $L_1, \ldots, L_m \in \mathbf{R}$, which are dependent on node coordinates x

(Cont'd)

- *Force mapping matrix* $P(\mathcal{X}) \in \mathbf{R}^{m \times 2n}$, which relates loads F to the internal stresses experienced by the bars, $f \in \mathbf{R}^m$; implicit in P is an adjacency matrix relating each bar to its attachment points
- *Stiffness matrix* $K(\mathcal{X}, a, L)$, which determines the amount of flex in the truss

$$K = \sum_{i=1}^{m} \frac{E_i a_i}{L_i^2} p_i p_i^T,$$

where $p_1, \ldots, p_m$ are the columns of the force mapping matrix P

Our truss design optimization is further characterized by the following variables, whose relations contain all of the physics of the problem:

- *Node deflections* $u \in \mathbf{R}^{2n}$ due to the truss flexing under loads F, where $u_j \in \mathbf{R}^2$ is the deflection of the j^{th} node; by Hooke's Law, we have the force balance $F = Ku$
- *Internal stress* $f_i \in \mathbf{R}$ experienced by each bar due to the node deflections

$$f_i = -\frac{E_i a_i}{L_i^2} p_i^T u, \quad i = 1, \ldots, m$$

- *Stored elastic energy* $\Theta = \frac{1}{2} F^T u$, which we minimize in order to maximize the truss stiffness

An Alternating Convex Optimization Approach

The minimization of Θ in (a, x) that follows from our formulation above is non-convex. As a heuristic to solve the optimization problem, we first optimize over the bar sizes a, and then over the node coordinates x:

- We perform a linear change of coordinates to cast the bar sizing problem as a second-order cone program (SOCP) in $w, v \in \mathbf{R}^m$:

$$w_i + v_i = -\frac{1}{2}\left(u^T P\right)_i f_i,$$

$$w_i - v_i = a_i$$

The value of $w_i + v_i$ is therefore the spring energy stored in the i^{th} bar

- Holding x constant, find the bar cross sectional areas a that minimize Θ:

$$\begin{array}{ll} \text{minimize} & \Theta = \mathbf{1}^T(w+v) \\ \text{subject to} & Pf + F = 0 \\ & M(w-v) \le d \\ & \left\| \left(v_i, \frac{L_i}{\sqrt{E_i}} f_i \right) \right\|_2 \le w_i, \quad i = 1, \ldots, m \\ & \mathbf{1}^T(w-v) \le W^{max} \end{array} \qquad (1)$$

- We then perform an affine change of coordinates to cast the node positioning problem as an SOCP in $w, v \in \mathbf{R}^m$ (different from above):

$$w_i + v_i = -\frac{1}{2}\left(u^T P\right)_i f_i,$$

$$w_i - v_i = \frac{2 p_i^T y_i}{L_i} + 1$$

(Cont'd)

- Holding a constant, find a set of displacements $y \in \mathbf{R}^{2n}$ that "shift" the node coordinates x from their original positions and minimize Θ:

$$\begin{array}{ll} \text{minimize} & \Theta = \mathbf{1}^T(w+v) \\ \text{subject to} & Pf + F = 0 \\ & \frac{1}{2}((w_i - v_i) - 1) = \frac{p_i^T y_i}{L_i}, \quad i = 1, \ldots, m \\ & \left\| \left(v_i, \frac{L_i}{\sqrt{E_i a_i}} f_i \right) \right\|_2 \le w_i, \quad i = 1, \ldots, m \\ & \|y_i\|_2 \le \epsilon_i, \quad i = 1, \ldots, m \\ & g(y) = 0, \end{array} \qquad (2)$$

where $g(y) = 0$ enforces truss symmetry, and $\|y_i\|_2 \le \epsilon_i$ restrict node shifts.

In our heuristic, we first discretize the physical space as in traditional approaches to obtain $\hat{\mathcal{D}}$, and alternate between solving (1) and (2) in each iteration:

given $\mathcal{X}^{\text{fixed}}$, $\mathcal{F}$, $\hat{\mathcal{D}}$

Generate set of node coordinates x^0 from $\hat{\mathcal{D}}$, set $x := x^0$

repeat

1. Given x, obtain a and Θ_1 as the solution to and objective of (1)
2. Given a, obtain y and Θ_2 as the solution to and objective of (2), set $x := x + y$
3. **break if** Θ_1 and Θ_2 converge

return a, x

Example: Bridge Design

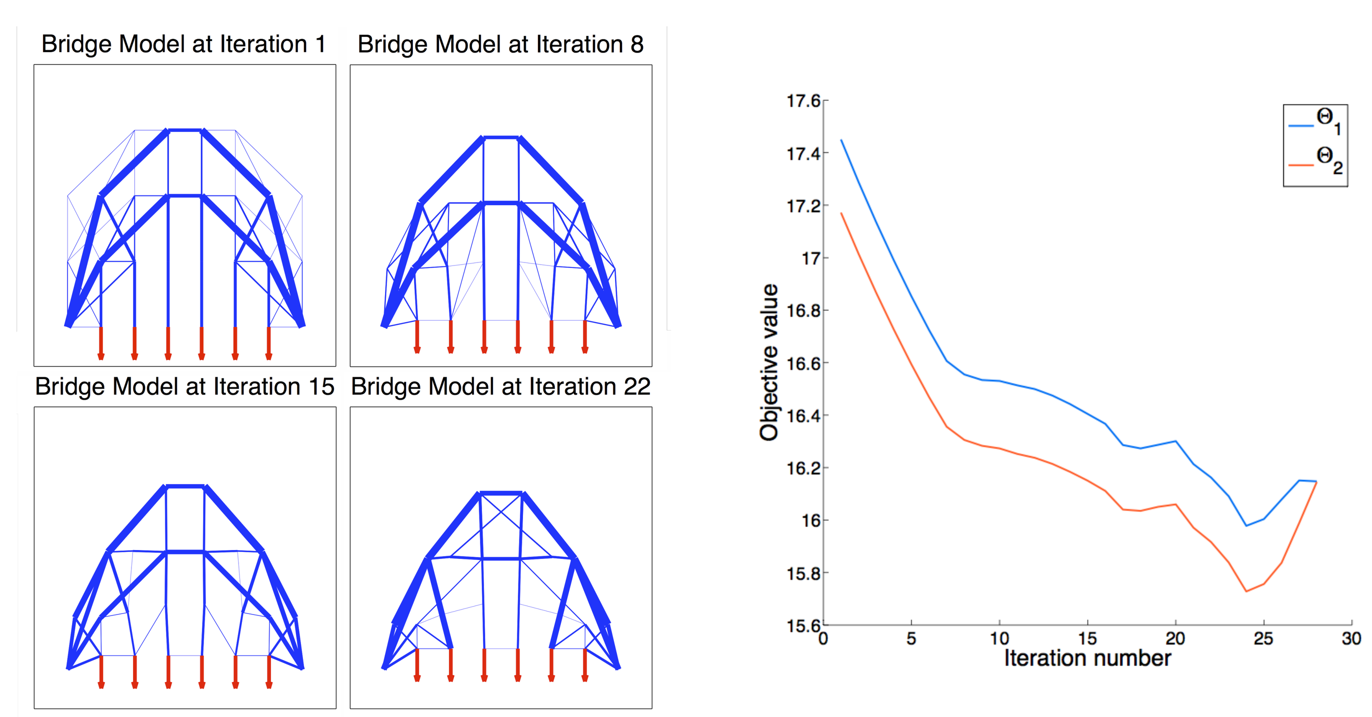

Figure 2: Left: Results of the iterations–notice the raised roadbed, reduced number of bars, and convergence to inverted catenary. Right: Convergence of our objective values.

This problem has 791 variables and 219 constraints. Out of several solvers, SCS was the fastest at 0.3 s per iteration (vs. 2.0 s with SeDuMi at comparable accuracy requirements). SCS's speed advantage scales with problem size (~100 times faster than SeDuMi with 16000 variables, 4000 constraints).

Conclusion

Our alternating convex optimization approach presents a promising tool to solving the non-convex truss design problem. Future work should extend the model to 3-D and compare this approach to other existing methods.

Acknowledgments

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